TUT 5

System of ODE,

$x_i = x_i(t)$

$$(1)\ \begin{cases} x_1' = F_1(t, x_1, x_2 \dots x_n) \\ x_2' = F_2(t, x_1, x_2 \dots x_n) \\ \vdots \\ x_n' = F_n(t, x_1, x_2 \dots x_n) \end{cases}$$

uniqueness and existence thm

Suppose each F_i, $\frac{\partial F_i}{\partial x_j}$ $i, j = 1, \dots n$ are Cts in region R of $tx_1 \dots x_n$ space s.t. $\alpha < t < \beta$, $\alpha_i < x_i < \beta_i$, let $(t_0, x_1^0, x_2^0, \dots, x_n^0)$ be in R, then $\exists$ an interval $|t - t_0| < h$ in which $\exists!$ solution $x_i = \phi_i(t)$.

We consider the n first order linear equation has form

$$(2)\ \begin{cases} x_1' = p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t) \\ x_2' = p_{21}(t)x_1 + \dots + p_{2n}(t)x_n + g_2(t) \\ \vdots \\ x_n' = p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t) \end{cases}$$

It is homogenous if all $g_i(t) = 0$, o.w. it is nonhomogen

If we consider the system of ②, then we the condition in uniqueness and existence thm change from $\left[F_i,\ \frac{\partial F_i}{\partial x_j} \text{ are } c\text{'s} \right]$ to $\left[P_{ij},\ g_i \text{ are } c\text{'s} \right]$

Actually, a n order ODE can be written as a system, Consider,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(t)$$

let

$$\begin{cases} x_1 = y \\ x_2 = y' = x_1' \\ \vdots \\ x_n = y^{(n-1)} = x_1^{(n-1)} \\ a_n x_n' = -a_{n-1} x_n - \cdots - a_0 x_1 + g(t). \end{cases}$$

for ②, we write it as

$$\vec{x}' = P(t)\ \vec{x} + \vec{g}(t)$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\quad P(t) = \begin{pmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & & \\ P_{n1} & \cdots & P_{nn} \end{pmatrix},\quad g(t) = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

Consider the homogenous case with constant P_{ij}, $P(t) = A$, $\vec{g}(t) = 0$

$$\vec{x}' = A\vec{x}$$

Similarly to $x' = ax$

$$x = Ce^{at}$$

Step ① find the eigenvalue of $A \in M^{2\times 2}(\mathbb{R})$

let them be r_1 and r_2, $r_1 \neq r_2$, real,

Step ② find the eigenvector corresponding to r_1; r_2, let them be $\vec{\xi}_1$ and $\vec{\xi}_2$,

Step ③, $\vec{x} = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t}$ will be a solution.

If $r_1 = u_1 + v_1 i$, $r_2 = \overline{r_1} = u_1 - v_1 i$

Step ② find the eigenvector corresponding to r_1, r_2, let them be $\vec{\xi}_1 = \vec{a}_1 + \vec{b}_1 i$, $\vec{\xi}_2 = \overline{\vec{\xi}_1} = \vec{a}_1 - \vec{b}_1 i$

Step ③, Consider $\vec{\xi}_1 \cdot e^{r_1 t} = \vec{\xi}_1 e^{u_1 t} \cdot e^{v_1 t i}$

$$= \vec{w} + \vec{z} i$$

then $\vec{x} = c_1\vec{w} + c_2\vec{z}$ is a solution.

Same result will be obtained if you use

$\vec{\xi}_2 \cdot e^{r_2 t}$.

If $r_1 = r_2 \in \mathbb{R}$, and A is not self-adjoint,

Step ② find the eigenvector of r_1, let it be ξ_1

Step ③, it is different from the 1-dim case, another solution is not $\xi_1 t e^{r_1 t}$ X!!!!

let $x^{(1)}(t) = \vec{\xi}_1 e^{r_1 t}$,

$x^{(2)}(t) = \vec{\xi}_1 t e^{r_1 t} + \vec{\eta} e^{r_1 t}$

Comb it to $\vec{X}' = A\vec{X}$,

we will have $t e^{r_1 t}(r_1 \vec{\xi}_1 - A\vec{\xi}_1) + e^{r_1 t}(\vec{\xi}_1 + r_1\vec{\eta} - A\vec{\eta}) = 0$

$$\therefore \begin{cases} r_1 \vec{\xi}_1 - A\vec{\xi}_1 = 0 \\ \vec{\xi}_1 + r_1 \vec{\eta} - A\vec{\eta} = 0 \end{cases}$$

we solve $\vec{\eta}$,

so $\vec{X} = \vec{X}^{(1)} + \vec{X}^{(2)}$.

Problem:

① $x' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} x$

Ans: let r be evalues

$$\begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = 0$$

$$(r-1)(r+4)+6=0$$

$$r=-2 \text{ or } -1$$

If $r=-2$, let $\vec{\xi}_1$ be its evector,

$$\begin{pmatrix} 1+2 & -2 \\ 3 & -4+2 \end{pmatrix} \begin{pmatrix} \xi_1^1 \\ \xi_1^2 \end{pmatrix} = 0$$

Consider $\begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix}$

$$3\xi_1^1 - 2\xi_1^2 = 0$$

$\vec{\xi}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

If $r=-1$, $\vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

So $\vec{x} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$.

② $\vec{x}' = \begin{pmatrix}1 & 2\\ -5 & -1\end{pmatrix}\vec{x}$,

Ans: $\begin{vmatrix}1-r & 2\\ -5 & -1-r\end{vmatrix} = 0$

$$(r-1)(r+1)+10=0$$

$$r=\pm 3i$$

If $r_1=3i$, let $\vec{\xi}_1$ be the eigenvector,

$$\begin{pmatrix}1-3i & 2\\ -5 & -1-3i\end{pmatrix} \sim \begin{pmatrix}5-15i & 10\\ -5+15i & (-1-3i)(1-3i)\end{pmatrix}$$

$$\sim \begin{pmatrix}0 & 0\\ -5+15i & -10\end{pmatrix} \sim \begin{pmatrix}0 & 0\\ -1+3i & -2\end{pmatrix}$$

So $\vec{\xi}_1 = \begin{pmatrix}-1-3i\\ 5\end{pmatrix}$,

So consider $\begin{pmatrix}-1-3i\\ 5\end{pmatrix} e^{0t}(\cos 3t + i\sin 3t)$

$$=\left(\begin{pmatrix}-1\\ 5\end{pmatrix} + \begin{pmatrix}-3\\ 0\end{pmatrix} i\right)(\cos 3t + i\sin 3t)$$

$$=\left[\begin{pmatrix}-1\\ 5\end{pmatrix}\cos 3t + \begin{pmatrix}3\\ 0\end{pmatrix}\sin t\right] + i\left(\begin{pmatrix}-3\\ 0\end{pmatrix}\cos 3t + \begin{pmatrix}-1\\ 5\end{pmatrix}\sin 3t\right)$$

$$\therefore\ \vec{x} = c_1\left(\begin{pmatrix}-1\\ 5\end{pmatrix}\cos 3t + \begin{pmatrix}3\\ 0\end{pmatrix}\sin t\right) + c_2\left(\begin{pmatrix}-3\\ 0\end{pmatrix}\cos 3t + \begin{pmatrix}-1\\ 5\end{pmatrix}\sin 3t\right)$$

③ $x' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \vec{x}$

Ans: let r be evalues,

$$\begin{vmatrix} 3-r & 1 \\ -4 & -1-r \end{vmatrix} = 0$$

$$(r-3)(1+r)+4=0$$

$$r = 1 \text{ (repeated)}$$

the corresponding evectors are $\vec{\xi}_1$,

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

let $\vec{\eta}$ s.t. $\vec{x}_2 = \vec{\xi}_1 te^{rt} + \vec{\eta} e^{rt}$

sub x_2 into $x' = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}\vec{x} = A\vec{x}$

$$\vec{\xi}_1 e^{rt} + \vec{\xi}_1 rte^{rt} + \vec{\eta} re^{rt} = A\vec{\xi}_1 te^{rt} + A\vec{\eta} e^{rt}$$

$$te^{rt}(\vec{\xi}_1 r - A\vec{\xi}_1) + e^{rt}(\vec{\xi}_1 + \vec{\eta} r - A\vec{\eta}) = 0$$

$$\therefore \begin{cases} \vec{\xi}_1 r - A\vec{\xi}_1 = 0 \\ \vec{\xi}_1 + \vec{\eta} r - A\vec{\eta} = 0 \end{cases}$$

$$(A - Ir)\vec{\eta} = \vec{\xi}_1$$

$$\left(\begin{array}{cc|c} 3-1 & 1 & 1 \\ -4 & -1-1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & 1 \\ -4 & -2 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

$2x_1 + x_2 = 1$

$x_2 = 1 - 2x_1$

$\therefore \vec{\eta} = \begin{pmatrix} k \\ 1-2k \end{pmatrix}$

you can choose k to be all value of $\mathbb{R}$,

Choose $k = 0$,

$\vec{X} = \vec{X_1} + \vec{X_2}$

$= c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t + c_2 \begin{pmatrix} t \\ -2t + 1 \end{pmatrix} e^t$